

**Dawson College**  
**Mathematics Department**  
**Final Examination**  
**201-NYB-05 Calculus II - Science**  
**Tuesday, May 24, 2011**

**Student Name:** \_\_\_\_\_

**Student I.D. #:** \_\_\_\_\_

**Teacher:** \_\_\_\_\_

**Instructors: M. Chaubey, D. Dubrovsky, Y. Lamontagne, S. Muise, V. Ohanyan,  
J. Requeima, S. Shahabi, S. Soltuz, O. Veres, O. Zlotchevskaia**

**TIME: 9:30 – 12:30 (3 hours)**

**INSTRUCTIONS:**

- Print your name and student I.D. number in the space provided above.
- Attempt all questions.
- All questions are to be answered directly on the examination paper.
- Translation and regular dictionaries are permitted.

<b>Question #</b>	<b>Marks</b>
<b>1 (20)</b>	
<b>2 (4)</b>	
<b>3 (8)</b>	
<b>4 (4)</b>	
<b>5 (4)</b>	

1. Find each indefinite integral.

$$a) \int \frac{3x^2 + 4x + 4}{x^3 + x} dx$$

$$\frac{3x^2 + 4x + 4}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$$3x^2 + 4x + 4 = A(x^2 + 1) + (Bx + C)x$$

$$\text{if } x=0, \quad \boxed{4 = A}$$

$$\therefore 3x^2 + 4x + 4 = 4x^2 + 4 + Bx^2 + Cx$$

$$\therefore 3 = 4 + B \Rightarrow \boxed{B = -1}$$

$$\boxed{4 = C}$$

$$= \int \left( \frac{4}{x} + \frac{-x + 4}{x^2 + 1} \right) dx = 4 \int \frac{1}{x} dx - \int \frac{x}{x^2 + 1} dx + 4 \int \frac{1}{x^2 + 1} dx$$

$$= 4 \ln|x| - \frac{1}{2} \ln|x^2 + 1| + 4 \arctan x + C$$

$$\begin{aligned}
 \text{b) } \int \tan^3 x \, dx &= \int \tan^2 x \cdot \tan x \, dx \\
 &= \int (\sec^2 x - 1) \cdot \tan x \, dx \\
 &= \int \tan x \cdot \sec^2 x \, dx - \int \tan x \, dx \\
 &= \frac{\tan^2 x}{2} + \ln |\cos x| + C
 \end{aligned}$$

$$\begin{aligned}
 \text{c) } \int x^2 e^{4x} \, dx \quad & u = x^2 \quad du = 2x \, dx \\
 & du = 2x \, dx \quad v = \frac{1}{4} e^{4x}
 \end{aligned}$$

$$= \frac{x^2 e^{4x}}{4} - \frac{1}{2} \int x e^{4x} \, dx, \quad \begin{array}{l} u = x \\ du = dx \end{array} \quad \begin{array}{l} dv = e^{4x} \, dx \\ v = \frac{1}{4} e^{4x} \end{array}$$

$$= \frac{x^2 e^{4x}}{4} - \frac{1}{2} \left( \frac{x e^{4x}}{4} - \frac{1}{4} \int e^{4x} \, dx \right)$$

$$= \frac{x^2 e^{4x}}{4} - \frac{x e^{4x}}{8} + \frac{1}{8} \left( \frac{1}{4} e^{4x} \right) + C$$

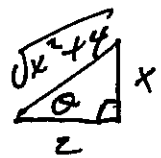
$$= \frac{x^2 e^{4x}}{4} - \frac{x e^{4x}}{8} + \frac{e^{4x}}{32} + C$$

$$d) \int \frac{1}{x^2 \sqrt{x^2+4}} dx$$

$$x = 2 \tan \theta$$

$$dx = 2 \sec^2 \theta d\theta$$

$$x^2 + 4 = 4 \sec^2 \theta.$$



$$= \int \frac{2 \sec^2 \theta d\theta}{4 \tan^2 \theta \cdot \sqrt{4 \sec^2 \theta}} = \frac{1}{4} \int \frac{\sec \theta}{\tan^2 \theta} d\theta$$

$$= \frac{1}{4} \int \frac{1}{\cos \theta} \cdot \frac{\cos^2 \theta}{\sin^2 \theta} d\theta = \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} d\theta$$

$u = \sin \theta$   
 $du = \cos \theta d\theta$

$$= \frac{1}{4} \int u^{-2} du = \frac{1}{4} \frac{u^{-1}}{-1} + C = \frac{-1}{4 \sin \theta} + C$$

$$= \frac{-1}{4 \left( \frac{x}{\sqrt{x^2+4}} \right)} + C = \frac{-\sqrt{x^2+4}}{4x} + C$$

$$e) \int \frac{dx}{2+\sqrt{x}}$$

$$u = 2 + \sqrt{x}$$

$$du = \frac{1}{2} dx$$

$$= \int \frac{2(u-2) du}{u}$$

$$2\sqrt{x} du = dx$$

$$2(u-2) du = dx$$

$$= 2 \int \left(1 - \frac{2}{u}\right) du = 2(u - 2 \ln|u|) + C$$

$$= 2(2 + \sqrt{x} - 2 \ln|2 + \sqrt{x}|) + C$$

2. Evaluate the definite integral  $\int_0^{\pi} \sin^2(3\theta) d\theta$ .

$$= \frac{1}{2} \int_0^{\pi} (1 - \cos(6\theta)) d\theta = \frac{1}{2} \left( \theta - \frac{1}{6} \sin(6\theta) \right) \Big|_0^{\pi}$$

$$= \frac{1}{2} \left[ \left( \pi - \frac{1}{6} \sin(6\pi) \right) - \left( 0 - \frac{1}{6} \sin(0) \right) \right]$$

$$= \pi/2.$$

its value.

$$a) \int_0^1 \frac{x+1}{\sqrt{x^2+2x}} dx$$

$$u = x^2 + 2x$$
$$du = (2x+2)dx$$
$$\frac{1}{2} du = (x+1)dx$$

$$x=0 \rightarrow u=0$$

$$x=1 \rightarrow u=3$$

$$= \frac{1}{2} \int_0^3 \frac{du}{\sqrt{u}}$$

$$= \frac{1}{2} \lim_{b \rightarrow 0^+} \int_b^3 u^{-1/2} du = \frac{1}{2} \lim_{b \rightarrow 0^+} \left( 2u^{1/2} \right) \Big|_b^3$$

$$= \lim_{b \rightarrow 0^+} (\sqrt{3} - \sqrt{b}) = \sqrt{3} \text{ (ans)}$$

4. Find the average value of the function  $f(x) = \ln x$  over the interval  $[1, e]$ .

$$C = \frac{1}{e-1} \int_1^e \ln x \, dx \quad \int \ln x \, dx = x \ln x - \int dx$$

$$= \frac{1}{e-1} (x \ln x - x) \Big|_1^e$$

$$= \frac{1}{e-1} [(e \ln e - e) - (1 \cdot \ln 1 - 1)]$$

$$= \frac{1}{e-1} [(e - e) - (-1)] = \frac{1}{e-1}$$

5. Evaluate  $\frac{d}{dx} \int_{\sec x}^2 \sqrt{1+t^3} \, dt$ .

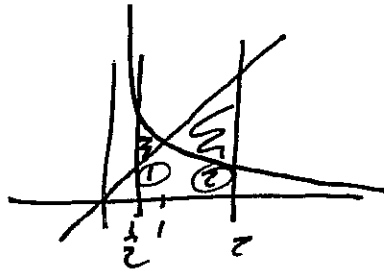
$$= \frac{d}{dx} \int_2^{\sec x} \sqrt{1+t^3} \, dt$$

6. Use the definition of the definite integral to calculate  $\int_2^4 (x^2 - 4x) dx$ . Verify

$$\Delta x = \frac{4-2}{2} \quad x_i = 2 + \frac{2i}{2}$$



7. Find the area of the region bounded by the graphs of  $y = x$  and  $y = \frac{1}{x^2}$ , between  $x = \frac{1}{2}$  and  $x = 2$ .

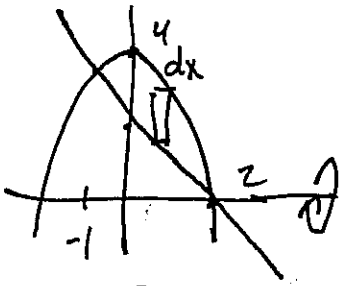


$$\begin{aligned} \textcircled{1} &= \int_{1/2}^1 \left( \frac{1}{x^2} - x \right) dx = \left( -\frac{1}{x} - \frac{x^2}{2} \right) \Big|_{1/2}^1 \\ &= \left[ \left( -1 - \frac{1}{2} \right) - \left( -2 - \frac{1}{8} \right) \right] = \frac{5}{8} \end{aligned}$$

$$\begin{aligned} \textcircled{2} &= \int_1^2 \left( x - \frac{1}{x^2} \right) dx = \left( \frac{x^2}{2} + \frac{1}{x} \right) \Big|_1^2 \\ &= \left[ \left( \frac{4}{2} + \frac{1}{2} \right) - \left( \frac{1}{2} + 1 \right) \right] = 1 \end{aligned}$$

$\frac{13}{8}$

8. Use disks or washers to find the volume of the solid obtained by revolving the region bounded by the graphs of  $y = 4 - x^2$  and  $y = 2 - x$  about the  $x$ -axis.



$$R(x) = 4 - x^2$$

$$r(x) = 2 - x$$

$$V = \pi \int_{-1}^2 [(4 - x^2)^2 - (2 - x)^2] dx$$

$$= \pi \int_{-1}^2 (16 - 8x^2 + x^4 - (4 - 4x + x^2)) dx$$

$$= \pi \int_{-1}^2 (12 + 4x - 9x^2 + x^4) dx$$

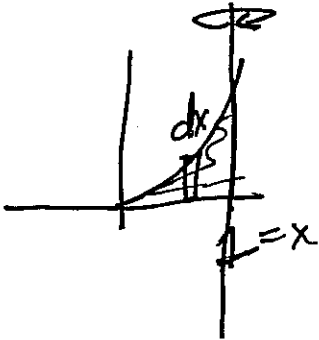
$$= \pi \left( 12x + 2x^2 - 3x^3 + \frac{x^5}{5} \right) \Big|_{-1}^2$$

$$= \pi \left[ \left( 24 + 8 - 24 + \frac{32}{5} \right) - \left( -12 + 3 - \frac{1}{5} \right) \right]$$

$$= \frac{108\pi}{5}$$

$$\frac{108\pi}{5}$$

9. Use shells to find the volume of the solid obtained by revolving the region bounded by the graphs of  $y = 0$  and  $y = 3x^4$  between  $x = 0$  and  $x = 1$  about



$$p(x) = 1 - x$$

$$h(x) = 3x^4.$$

$$V = 2\pi \int_0^1 (1-x)(3x^4) dx$$

$$= 2\pi \int_0^1 (3x^4 - 3x^5) dx$$

$$= 2\pi \left( \frac{3x^5}{5} - \frac{x^6}{2} \right) \Big|_0^1$$

$$= 2\pi \left( \frac{3}{5} - \frac{1}{2} \right) = ~~2\pi \left( \frac{1}{10} \right)~~ = \frac{\pi}{5}$$

10. Find the arclength of the curve defined by the function  $f(x) = \frac{x^3}{12} + \frac{1}{x}$  over the interval  $[1, 2]$ .

$$\begin{aligned}
 1 + (f'(x))^2 &= 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4} \\
 &= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2
 \end{aligned}$$

$$\therefore s = \int_1^2 \sqrt{1 + f'(x)^2} dx = \int_1^2 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$$

$$= \left(\frac{x^3}{12} - \frac{1}{x}\right) \Big|_1^2 = \left(\frac{8}{12} - \frac{1}{2}\right) - \left(\frac{1}{12} - 1\right) = \frac{13}{12}$$

11. Use the squeeze theorem to find the limit of the sequence  $\left\{\frac{2^n}{n!}\right\}$ , if it exists.

$$\begin{aligned}
 0 < \frac{2^n}{n!} &= \frac{2 \cdot \cancel{2} \cdot 2 \cdot \dots \cdot 2}{1 \cdot \cancel{2} \cdot 3 \cdot \dots \cdot n} = (2) \cdot \underset{<1}{\frac{2}{3}} \cdot \underset{<1}{\frac{2}{4}} \cdot \dots \cdot \underset{<1}{\left(\frac{2}{n-1}\right)} \cdot \frac{2}{n} \\
 &< 2 \left(\frac{2}{n}\right) = \frac{4}{n} \rightarrow 0 \text{ as } n \rightarrow \infty
 \end{aligned}$$

0 by squeeze theorem

12. Find the sum of each series, if it converges, or show that it diverges.

a)  $\sum_{n=1}^{\infty} \frac{3^n - 1}{6^{n-1}}$

~~$\sum_{n=1}^{\infty} \frac{3^n - 1}{6^{n-1}}$~~

$$= \sum_{n=1}^{\infty} \frac{3^n}{6^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$

$r = 1/2 < 1 \checkmark$        $r = 1/6 < 1 \checkmark$

$a = 3$        $a = 1$

$$= \frac{3}{1 - 1/2} - \frac{1}{1 - 1/6}$$

$$= \frac{3}{1/2} - \frac{1}{5/6} = 6 - \frac{6}{5} = \frac{24}{5}$$

b)  $\sum_{n=2}^{\infty} \ln\left(\frac{n+1}{n+2}\right) = \sum_{n=0}^{\infty} (\ln(n+1) - \ln(n+2))$

$$S_n = (\ln(3) - \ln(4)) + (\ln(4) - \ln(5)) + \dots + (\ln(n) - \ln(n+1)) + (\ln(n+1) - \ln(n+2))$$

$$S_n = \ln 3 - \ln(n+2)$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \ln 3 - \infty = -\infty \Rightarrow \text{div}$$

13. Determine whether each series converges or diverges. State any tests used to reach your conclusions.

a)  $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

$$\frac{\sin^2 n}{2^n} \leq \frac{1}{2^n} \in \text{geo with } r = \frac{1}{2} < 1$$

CONV

$\therefore$  CONV by DCT.

b)  $\sum_{n=1}^{\infty} \cos(n\pi)$

$$\lim_{n \rightarrow \infty} \cos(n\pi) = \text{DNE}$$

$\Rightarrow$  div. by Div Test.

$$c) \sum_{n=2}^{\infty} \frac{1}{n \sqrt[3]{\ln n}}$$

$$f(x) = \frac{1}{x \sqrt[3]{\ln x}}$$

pos, cont.  
decreasing since  
1 over increasing

$$\int_2^{\infty} \frac{1}{x (\ln x)^{1/3}} dx = \lim_{b \rightarrow \infty} \int_{\ln 2}^b u^{-1/3} du$$

$$u = \ln x \\ du = \frac{1}{x} dx$$

$$= \lim_{b \rightarrow \infty} \left( \frac{3}{2} u^{2/3} \right) \Big|_{\ln 2}^b$$

$$= \lim_{b \rightarrow \infty} \left( b^{2/3} - (\ln 2)^{2/3} \right) = \infty \Rightarrow \text{div}$$

by integral  
test.

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$$d) \sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$$

$$a_{n+1} = \frac{(n+1)!}{(2(n+1)+1)!} = \frac{(n+1) \cdot n!}{(2n+3)(2n+2) \cdot (2n+1)!}$$

$$\therefore \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1) \cdot n!}{(2n+3)(2n+2) \cdot (2n+1)!} \cdot \frac{(2n+1)!}{n!}$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{(2n+3)(2n+2)} = 0 \Rightarrow \text{conv.}$$

ratio test.

14 Determine if the series  $\sum_{n=1}^{\infty} (-1)^n$  converges absolutely.

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16. Find the radius of convergence and the interval of convergence of the power

series  $\sum_{n=0}^{\infty} \frac{(x-2)^n}{\sqrt{n+1}}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-2)^n} \right|$$

$$= |x-2| \cdot \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}} = |x-2| < 1$$

$$1 < x < 3$$

radius = 1.

if  $x=1$ ,  $= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$  conv. by A.S.T.  
( $b_n = \frac{1}{\sqrt{n}}$ ,  $b_{n+1} < b_n$ ,  $\lim_{n \rightarrow \infty} b_n = 0$ ).

if  $x=3$ ,  $= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  div p-series.

$\therefore$  interval =  $(1, 3)$ .