

201-NYA-05
Calculus 1 (Science)
Final Exam

May 16, 2011

Name: SOLUTIONS

Student ID: _____

- There are a total of 100 marks on this test.

- Show all your work and answers in the provided space.

1. [4+4+4 marks] Find each of the following limits. Please show all your work and give exact

answers (no decimals).

$$(a) \lim_{x \rightarrow 2} \frac{x^2 - 9x + 14}{3x^2 - 6x} \quad (b) \lim_{x \rightarrow 1} \frac{\sqrt{x^2 + 48} - 7}{x - 1}, \quad (c) \lim_{x \rightarrow 2} \frac{17x^5 - 3x^8}{6}$$

$$a) \lim_{x \rightarrow 2} \frac{x^2 - 9x + 14}{3x^2 - 6x} = \lim_{x \rightarrow 2} \frac{(x-7)(x-2)}{3x(x-2)} = \frac{2-7}{3(2)} = -\frac{5}{6}$$

or

$$\lim_{x \rightarrow 2} \frac{x^2 - 9x + 14}{3x^2 - 6x} = \lim_{x \rightarrow 2} \frac{2x - 9}{6x - 6} = \frac{4 - 9}{12 - 6} = -\frac{5}{6}$$

$$b) \lim_{x \rightarrow 1} \frac{\sqrt{x^2 + 48} - 7}{x - 1} \cdot \frac{\sqrt{x^2 + 48} + 7}{\sqrt{x^2 + 48} + 7} = \lim_{x \rightarrow 1} \frac{(x^2 + 48) - 49}{(x-1)(\sqrt{x^2 + 48} + 7)}$$
$$= \lim_{x \rightarrow 1} \frac{x^2 - 1}{(x-1)(\sqrt{x^2 + 48} + 7)} = \lim_{x \rightarrow 1} \frac{x+1}{\sqrt{x^2 + 48} + 7} = \frac{1+1}{\sqrt{49} + 7}$$
$$= \frac{2}{14} = \frac{1}{7}$$

or

2. [5 marks] Find the values of x where the following function is discontinuous. For each value, clearly state *why* $f(x)$ is discontinuous by using the definition of continuity at a point.

$$f(x) = \begin{cases} 1/x & \text{if } x < 1 \\ x & \text{if } 1 \leq x < 2 \\ 3x^2 - 11x & \text{if } x \geq 2 \end{cases}$$

$x=0$: $f(0)$ DNE $\implies f(x)$ discontinuous at $x=0$

$x=2$: $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} x = 2$

$$\begin{aligned} \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} 3x^2 - 11x = 3(4) - 11(2) \\ &= 12 - 22 = -10 \end{aligned}$$

$\lim_{x \rightarrow 2} f(x)$ DNE $\implies f(x)$ discontinuous at $x=2$.

3. [4+4+4+4 marks] Find the first derivative of each function. Answers do not need to be simplified but should be expressed only in terms of x

(a) $y = \tan(\sin x) + \tan x \sin x$

$$y' = \sec^2(\sin x) \cdot \cos x + [(\sec^2 x) \sin x + \tan x (\cos x)]$$

(b) $y = \frac{\ln(x^2 + x + 1)}{x^2 + x + 1}$

$$y' = \frac{\frac{1}{x^2 + x + 1} (2x + 1)(x^2 + x + 1) - \ln(x^2 + x + 1) \cdot (2x + 1)}{(x^2 + x + 1)^2}$$

$$(c) y = \cot^2(3x^2 + 5x)$$

$$y = (\cot(3x^2 + 5x))^2$$

$$\frac{1}{y} \frac{dy}{dx} = 2 \cot(3x^2 + 5x) \cdot (-\csc^2(3x^2 + 5x)) \cdot (6x + 5)$$

$$(d) y = (\sin x)^{3x}$$

$$\ln y = \ln((\sin x)^{3x})$$

$$\ln y = 3x \ln(\sin x)$$

$$\frac{1}{y} y' = 3 \ln(\sin x) + 3x \cdot \frac{1}{\sin x} (\cos x)$$

$$\frac{1}{y} y' = (3 \ln(\sin x) + 3x \cos x)$$

4. [5 marks] Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \frac{2}{x}$. No marks will be given for using derivative rules.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{2(x) - 2(x+h)}{(x+h)(x)} \cdot \frac{1}{h} \\ &= \lim_{h \rightarrow 0} \frac{2x - 2x - 2h}{xh(x+h)} \\ &= \lim_{h \rightarrow 0} \frac{-2}{x(x+h)} \\ &= \frac{-2}{x^2} \end{aligned}$$

5. [4 marks] Find the slope of the tangent line to the curve $x^2y - xy^2 = 6$ at the point $(3, 1)$.

$$\begin{aligned} x^2y - xy^2 &= 6 \\ (2xy + x^2y') - (1 \cdot y^2 + x \cdot 2yy') &= 0 \\ 2(3)(1) + (3)^2y' - (1)(1)^2 - (3)(2)(1)y' &= 0 \end{aligned}$$

$$\begin{aligned} 3y' &= -5 \\ y' &= -\frac{5}{3} \end{aligned}$$

6. [4+2 marks] Suppose $f(x) = \arcsin 2x - \sqrt{1-4x^2}$.

(a) Find $f'(x)$ and simplify.

(b) Find $f''(0)$.

$$a) f(x) = \arcsin(2x) - (1-4x^2)^{1/2}$$

$$f'(x) = \frac{1}{\sqrt{1-(2x)^2}}(2) - \frac{1}{2}(1-4x^2)^{-1/2}(-8x)$$

$$= \frac{2}{\sqrt{1-4x^2}} + \frac{8x}{2\sqrt{1-4x^2}}$$

$$= \frac{2+4x}{\sqrt{1-4x^2}}$$

$$b) f''(x) = (4)(1-4x^2)^{1/2} - (2+4x)\frac{1}{2}(1-4x^2)^{-1/2}(-8x)$$

$$f''(0) = \frac{(4)(1)^{1/2} - (2)(\frac{1}{2})(1)^{-1/2}(0)}{1-4(0)^2}$$

$$= \frac{4-0}{1}$$

$$= 4$$

7. [4+3 marks] Consider the function $g(x) = (3x - 4)^4(4x - 3)^3$.

(a) Find $g'(x)$ and simplify.

(b) Find the absolute maximum and minimum values of $g(x)$ on the interval $[0, 1]$.

$$\begin{aligned} \text{a) } g'(x) &= 4(3x-4)^3(3)(4x-3)^3 + 3(4x-3)^2(4)(3x-4)^4 \\ &= 12(3x-4)^3(4x-3)^2 \left[(4x-3) + (3x-4) \right] \\ &= 12(3x-4)^3(4x-3)^2(7x-7) \end{aligned}$$

b) Critical numbers:

$$0 = 12(3x-4)^3(4x-3)^2(7x-7)$$

$$\Rightarrow x = \frac{4}{3}, \frac{3}{4}, 1$$

On the interval $[0, 1]$, we check $x=0, \frac{3}{4}, 1$ only:

$$f(0) = (0-4)^4(0-3)^3 = (256)(-27) = -6912$$

$$f\left(\frac{3}{4}\right) = \left(\frac{9}{4}-4\right)^4(3-3)^3 = 0.$$

$$f(1) = (3-4)^4(4-3)^3 = (-1)^4(1)^3 = 1$$

Absolute maximum : 1

Absolute minimum : -6912

8. [5 marks] Boyles Law states that the product of the pressure exerted by a gas, P , and its volume V are related by $PV = k$ where k is a constant. At a point in time the volume of

gas is 2400 cm^3 and the pressure is 400 kPa . Find the value of k .

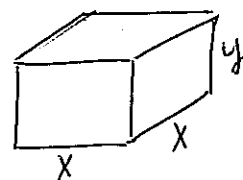
if the volume is decreasing at a rate of $570 \text{ cm}^3/\text{s}$ find the rate of change of pressure.

9. [7 marks] A box with a square base is to be constructed with volume 20 m^3 . The material for the base of the box costs $\$0.30/\text{m}^2$, the material for the sides costs $\$0.10/\text{m}^2$ and the

material for the top of the box costs $\$0.20/\text{m}^2$. Find the dimensions that minimize the cost of the box (answer with the appropriate units).

Minimize cost:

$$C = (0.30)x^2 + (0.10)(4xy) + (0.20)x^2$$



$$C = 0.5x^2 + 0.4xy$$

$$\text{Since } V = x^2y \Rightarrow xy = \frac{20}{x^2}$$

$$\begin{aligned} C(x) &= 0.5x^2 + 0.4x \left(\frac{20}{x^2} \right) \\ &= 0.5x^2 + \frac{8}{x} \end{aligned}$$

$$\Rightarrow C'(x) = x - \frac{8}{x^2}$$

$$0 = x - \frac{8}{x^2}$$

$$\frac{8}{x^2} = x$$

$$x^3 = 8$$

$$x = 2$$

The dimensions are

$$x = 2 \text{ m}, \quad y = \frac{20}{(2)^2} = 5 \text{ m}$$

Check max/min:

$$C''(x) = 1 + \frac{16}{x^3}$$

min at $x = 2$

10. [2+3+3+2 marks] Consider the following function and its derivatives:

$$f(x) = \frac{4(1-x)}{x^2}$$

$$f'(x) = \frac{4(x-2)}{x^3}$$

$$f''(x) = \frac{-8(x-3)}{x^4}$$

- Find the vertical and horizontal asymptotes, if they exist.
- Find the intervals of increase & decrease and local maxima & minima, if they exist.
- Find the intervals of concavity and inflection points, if they exist.
- Sketch the graph of $f(x)$.

NOTE: A grid and extra space may be found on the next page.

a) V. asymptotes: $\lim_{x \rightarrow 0} \frac{4(1-x)}{x^2} = \infty \quad \left(\frac{+}{0^+}\right)$

H. asymptote: $\lim_{x \rightarrow \pm\infty} \frac{4-4x}{x^2} = \lim_{x \rightarrow \pm\infty} \frac{4}{x^2} - \frac{4}{x} = 0$

\Rightarrow VA at $x=0$, HA at $y=0$

b)

$$4(x-2) \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ | \\ 2 \end{array} \begin{array}{c} + \\ + \end{array}$$

$$x^3 \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ | \\ 0 \end{array} \begin{array}{c} + \\ + \\ + \\ + \\ + \end{array}$$

$$f'(x) \begin{array}{c} + \quad \text{---} \quad - \quad \text{---} \quad + \\ | \quad \quad | \\ 0 \quad \quad 2 \end{array}$$

Increasing on $(-\infty, 0) \cup (2, \infty)$
 Decreasing on $(0, 2)$

$$f(2) = \frac{4(1-2)}{(2)^2} = -1 \Rightarrow \text{local minimum at } (2, -1)$$

no local maximum

Extra space for Question #10.

$$f(x) = \frac{4(1-x)}{x^2}$$

$$f'(x) = \frac{4(x-2)}{x^3}$$

$$f''(x) = \frac{-8(x-3)}{x^4}$$

c)

$$\begin{array}{c} -8(x-3) \\ x^4 \\ f''(x) \end{array} \quad \begin{array}{c} \frac{+++++}{3} \quad \frac{--}{3} \\ \frac{++}{0} \quad \frac{++++}{0} \\ \frac{+}{0} \quad \frac{+}{3} \quad \frac{-}{3} \end{array}$$

Concave up: $(-\infty, 0) \cup (0, 3)$

Concave down: $(3, \infty)$

11. [4+4 marks] Find each limit:

$$(a) \lim_{x \rightarrow \infty} x^3 e^{-x^2}$$

$$a) \lim_{x \rightarrow \infty} x^3 e^{-x^2} \quad (\infty \cdot 0)$$

$$= \lim_{x \rightarrow \infty} \frac{x^3}{e^{x^2}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \rightarrow \infty} \frac{3x^2}{2x e^{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{3x}{2e^{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{3}{2(2x)e^{x^2}}$$

$$= 0.$$

$$(b) \lim_{x \rightarrow 0} (1-x^2)^{1/x^2}$$

$$b) L = \lim_{x \rightarrow 0} (1-x^2)^{1/x^2}$$

$$\ln L = \lim_{x \rightarrow 0} \ln(1-x^2)^{1/x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\ln(1-x^2)}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{1-x^2}(-2x)}{2x}$$

$$= \frac{-1}{1-0} = -1$$

$$\Rightarrow L = e^{-1} = \frac{1}{e}$$

12. [3 marks] The velocity of a particle (in m/s) is given by the function $v(t) = \sin t - 4\sqrt{t}$. Find the position function $s(t)$ if $s(0) = 10$ m.

$$v(t) = \sin t - 4t^{1/2}$$

$$= -\cos t - \frac{8}{3}t^{3/2} + C$$

$$10 = -\cos(0) - \frac{8}{3}(0)^{3/2} + C$$

$$10 = -1 - 0 + C$$

$$\Rightarrow C = 11$$

13. [4 marks] Solve the following separable differential equation:

$$(1+x^2)\frac{dy}{dx} = 2y$$

$$\int \frac{1}{y} dy = \int \frac{2}{1+x^2} dx$$

$$\ln|y| = 2 \arctan x + C$$

14. [4+4 marks] Evaluate each integral.

(a) $\int (x^3 + 6x^2 - 4)^8 (6x^2 + 24x) dx$

(b) $\int \frac{5 \cos(\ln x)}{3x} dx$

a) $u = x^3 + 6x^2 - 4$
 $du = (3x^2 + 12x) dx$
 $2 du = (6x^2 + 24x) dx$

$$\begin{aligned} \int (x^3 + 6x^2 - 4)^8 (6x^2 + 24x) dx &= \int u^8 \cdot 2 du \\ &= 2 \int u^8 du \\ &= \frac{2}{9} u^9 + C \\ &= \frac{2}{9} (x^3 + 6x^2 - 4)^9 + C \end{aligned}$$

b) $u = \ln x$
 $du = \frac{1}{x} dx$

$$\begin{aligned} \int \frac{5 \cos(\ln x)}{3x} dx &= \frac{5}{3} \int \cos(\ln x) \frac{1}{x} dx \\ &= \frac{5}{3} \int \cos u du \\ &= \frac{5}{3} \sin u + C \\ &= \frac{5}{3} \sin(\ln x) + C \end{aligned}$$