

CLASS NOTES

SYNTHETIC DIVISION

When factoring or evaluating polynomials we often find that it is convenient to divide a polynomial by a linear (first degree) binomial of the form $x - k$ where k is a real number. In certain problems we must use trial-and-error to find a particular value of k . The resulting process of repeated divisions can be tedious and time-consuming. The use of **synthetic division** (instead of long division) can save a lot of time and effort.

It must be emphasized that synthetic division may only be used when dividing a polynomial by a binomial of the form $x - k$.

Illustration: Suppose we wish to divide the polynomial $P(x) = 5x^3 - 6x^2 + 28x - 2$ by $x - 3$. The k -value is 3 (the number subtracted from x in the binomial $x - 3$). $P(x)$ is a polynomial of degree 3, and the four coefficients of $P(x)$ are 5, -6, 28, and -2.

The division may now be expressed as shown below. Note that there is a blank space above the horizontal line.

$$\begin{array}{r|rrrr} 3 & 5 & & -6 & 28 & -2 \\ \hline & & & & & \end{array}$$

We "bring down" (copy) the leading coefficient below the line in the same column.

$$\begin{array}{r|rrrr} 3 & 5 & & -6 & 28 & -2 \\ \hline & 5 & & & & \end{array}$$

We now multiply 5 (the number below the line in the first column) by the k -value (3) and write the result in the next column (second column) above the line.

$$\begin{array}{r|rrrr} 3 & 5 & & -6 & 28 & -2 \\ \hline & 5 & 15 & & & \end{array}$$

Then we add the two numbers that are above the line in the second column and write the sum below the line.

$$\begin{array}{r|rrrr} 3 & 5 & & -6 & 28 & -2 \\ \hline & 5 & 9 & & & \end{array}$$

We now multiply 9 (the number below the line in the second column) by the k -value (3) and write the result in the next column (third column) above the line.

$$\begin{array}{r|rrrr} 3 & 5 & 6 & 28 & 2 \\ & & 15 & 27 & \\ \hline & 5 & 9 & & \end{array}$$

Then we add the two numbers that are above the line in the third column and write the sum below the line.

$$\begin{array}{r|rrrr} 3 & 5 & 6 & 28 & 2 \\ & & 15 & 27 & \\ \hline & 5 & 9 & 1 & \end{array}$$

We repeat the previous two steps one last time to get the final result.

$$\begin{array}{r|rrrr} 3 & 5 & 6 & 28 & 2 \\ & & 15 & 27 & 3 \\ \hline & 5 & 9 & 1 & 5 \end{array}$$

The first three numbers below the line (5, 9, and -1) are the coefficients of the quotient. The last number below the line (-5) is the remainder R . We can therefore rewrite the division as the sum of a polynomial (the quotient) and a rational (the remainder divided by the divisor).

$$\frac{5x^3 + 6x^2 + 28x + 2}{x - 3} = 5x^2 + 9x - 1 + \frac{5}{x - 3}$$

NOTE: We can see that $P(3) = 5(3)^3 + 6(3)^2 + 28(3) + 2 = 5$. i.e. the value of $P(k)$ is equal to the remainder, R , when we divide $P(x)$ by $x - k$. More on this later (Remainder Theorem).

Example: Perform the division $\frac{3x^4 + 4x^2 + 9x - 6}{x + 2}$ and rewrite as a polynomial plus rational.

NOTE: The divisor may be written $x - (-2)$! $k = -2$. Furthermore, the numerator, which is a polynomial of degree 4, has five terms. The five terms are $3x^4 + 0x^3 + 4x^2 + 9x - 6$. We use synthetic division to find the quotient and remainder.

$$\begin{array}{r|rrrrr} 2 & 3 & 0 & 4 & 9 & 6 \\ & & 6 & 12 & 16 & 14 \\ \hline & 3 & 6 & 8 & 7 & 8 \end{array} \quad ! \quad \text{The quotient is } 3x^3 + 6x^2 + 8x + 7 \text{ and the remainder is } 8.$$

$$! \quad \frac{3x^4 + 4x^2 + 9x - 6}{x + 2} = 3x^3 + 6x^2 + 8x + 7 + \frac{8}{x + 2}$$

Exercises: Perform each division and rewrite as a polynomial plus a rational.

$$1. \frac{x^3 + 7x^2 + 15}{x + 4}$$

$$3. \frac{4x^4 - 7x^3 + 9x^2 - 3x + 8}{x - 1}$$

$$2. \frac{2x^4 + 7x^3 - x + 5}{x + 3}$$

$$4. \frac{3x^5 - 4x + 1}{x - 2}$$

Answers to Exercises:

$$1. \begin{array}{r|rrrr} 4 & 1 & 7 & 0 & 15 \\ & & 4 & 12 & 48 \\ \hline & 1 & 3 & 12 & 63 \end{array} \quad / \quad \frac{x^3 + 7x^2 + 15}{x + 4} = x^2 + 3x + 12 + \frac{63}{x + 4}$$

$$2. \begin{array}{r|rrrrr} 3 & 2 & 7 & 0 & 1 & 5 \\ & & 6 & 3 & 9 & 24 \\ \hline & 2 & 1 & 3 & 8 & 19 \end{array} \quad / \quad \frac{2x^4 + 7x^3 - x + 5}{x + 3} = 2x^3 + x^2 + 3x + 8 + \frac{19}{x + 3}$$

$$3. \begin{array}{r|rrrrr} 1 & 4 & 7 & 9 & 3 & 8 \\ & & 4 & 3 & 6 & 3 \\ \hline & 4 & 3 & 6 & 3 & 5 \end{array} \quad / \quad \frac{4x^4 - 7x^3 + 9x^2 - 3x + 8}{x - 1} = 4x^3 + 3x^2 + 6x + 3 + \frac{5}{x - 1}$$

$$4. \begin{array}{r|rrrrr} 2 & 3 & 0 & 0 & 0 & 4 & 1 \\ & & 6 & 12 & 24 & 48 & 88 \\ \hline & 3 & 6 & 12 & 24 & 44 & 89 \end{array} \quad / \quad \frac{3x^5 - 4x + 1}{x - 2} = 3x^4 + 6x^3 + 12x^2 + 24x + 44 + \frac{89}{x - 2}$$

REMAINDER THEOREM

At the end of the previous illustration we saw that $P(3) = 5$. Similarly, in the example we saw that $P(-2) = 8$. In both cases the value of the polynomial at $x = k$ is equal to the remainder R . The same principle applies in each of the exercises above. This principle illustrates the Remainder Theorem.

Remainder Theorem:

If the polynomial $P(x)$ is divided by $x - k$, then the remainder R is equal to $P(k)$.

$$\begin{array}{r|rrrrr}
 7 & 2 & 1 & 3 & 5 & 4 \\
 & & 14 & 91 & 616 & 4277 \\
 \hline
 & 2 & 13 & 88 & 611 & 4281
 \end{array} \quad ! \quad P(7) = 4281$$

Exercises: In each of the following evaluate the polynomial $P(x)$ at the given k -value.

- $P(x) = 2x^3 - 8x^2 + 4x + 5$ at $k = 6$
- $P(x) = 5x^4 - 3x^2 + 2x - 1$ at $k = 4$
- $P(x) = 6x^4 - x^3 + 9x^2 - 7$ at $k = 7$
- $P(x) = 4x^5 + 2x^4 - 5x^3 + 6x^2 - 3x + 8$ at $k = 5$

Answers to Exercises:

$$\begin{array}{r|rrrr}
 6 & 2 & 8 & 4 & 5 \\
 & & 12 & 24 & 120 \\
 \hline
 & 2 & 4 & 20 & 125
 \end{array} \quad ! \quad P(6) = 125$$

$$\begin{array}{r|rrrrr}
 4 & 5 & 0 & 3 & 2 & 1 \\
 & & 20 & 80 & 308 & 1224 \\
 \hline
 & 5 & 20 & 77 & 306 & 1223
 \end{array} \quad ! \quad P(4) = 1223$$

$$\begin{array}{r|rrrrr}
 7 & 6 & 1 & 9 & 0 & 7 \\
 & & 42 & 287 & 2072 & 14504 \\
 \hline
 & 6 & 41 & 296 & 2072 & 14497
 \end{array} \quad ! \quad P(7) = 14497$$

$$\begin{array}{r|rrrrr}
 5 & 4 & 2 & 5 & 6 & 3 & 8 \\
 & & 20 & 90 & 425 & 2095 & 10460 \\
 \hline
 & 4 & 18 & 85 & 419 & 2092 & 10468
 \end{array} \quad ! \quad P(5) = 10468$$

FACTOR THEOREM

We have seen that if a polynomial $P(x)$ is divided by $x - k$ to get the quotient $Q(x)$ and remainder R , then we can write $\frac{P(x)}{x - k} = Q(x) + \frac{R}{x - k}$. And we know, by the Remainder Theorem, that $P(k) = R$. It follows that if $P(k) = 0$ then $R = 0$ and $\frac{P(x)}{x - k} = Q(x)$. This last equation may be rewritten $P(x) = (x - k) Q(x)$ which means that $x - k$ is a factor of $P(x)$.

Factor Theorem:

The polynomial $x - k$ is a factor of polynomial $P(x)$ if and only if $P(k) = 0$.

Example: Given that $P(-3) = 0$ rewrite the polynomial $P(x) = 6x^3 + 19x^2 + 2x - 3$ as the product of two factors.

Solution: By the Factor Theorem we know that $x + 3$ is one factor of $P(x)$. To get the other factor we can use synthetic division to divide $P(x)$ by $x + 3$ (keeping in mind that $k = -3$).

$$\begin{array}{r|rrrr}
 3 & 6 & 19 & 2 & 3 \\
 & & 18 & 3 & 3 \\
 \hline
 & 6 & 1 & 1 & 0
 \end{array}
 \quad ! \quad P(x) = (x + 3)(6x^2 + x + 1)$$

Exercises: In each of the following rewrite the polynomial $P(x)$ as the product of two factors.

1. $P(x) = x^4 + 2x^3 - 7x^2 - 20x + 12$ given $P(-2) = 0$

2. $P(x) = 2x^5 + x^4 - 30x^3 - 35x^2 + 38x + 24$ given $P(4) = 0$

Answers to Exercises:

1.
$$\begin{array}{r|rrrrr}
 2 & 1 & 2 & 7 & 20 & 12 \\
 & & 2 & 0 & 14 & 12 \\
 \hline
 & 1 & 0 & 7 & 6 & 0
 \end{array}
 \quad ! \quad P(x) = (x + 2)(x^3 - 7x + 6)$$

2.
$$\begin{array}{r|rrrrrr}
 4 & 2 & 1 & 30 & 35 & 38 & 24 \\
 & & 8 & 36 & 24 & 44 & 24 \\
 \hline
 & 2 & 9 & 6 & 11 & 6 & 0
 \end{array}
 \quad ! \quad P(x) = (x - 4)(2x^4 + 9x^3 + 6x^2 + 11x + 6)$$

FACTORIZING POLYNOMIALS OF HIGHER DEGREE (degree higher than 2)

Terminology: A **zero** of a polynomial P is a number k such that $P(k) = 0$ (we worked with "zeros" in the preceding section). A zero of P is called a **root** or **solution** of the polynomial $P(x) = 0$. Clearly the technical difference between "zeros" and "roots" is of little importance to us. Furthermore, the term "zero" can be confusing to the student (a "zero" of the polynomial is not necessarily equal to zero). Therefore, in the remainder of this discussion the term "root" will be used exclusively.

The Rational Roots Test:

Suppose the polynomial $P(x)$ has integer coefficients. If the rational number $\frac{n}{d}$ (in lowest terms) is a root of P then the numerator n is a factor of the constant term of P and the denominator d is a factor of the leading coefficient.

The Integer Roots Test:

Suppose the polynomial $P(x)$ has integer coefficients with leading coefficient 1. If the integer k is a root of P then k is a factor of the constant term of polynomial P .

The Rational Roots Test and the Integer Roots Test can only give us *possible* roots of a given polynomial. We must test these possible values (using synthetic division) to determine which, if any, of the possibilities are *actual* roots. Once we have the actual root(s) then we can write the polynomial in factored form.

Example 1: Completely factor the polynomial $P(x) = x^3 - 5x^2 - 2x + 24$.

Solution: Polynomial P has integer coefficients with leading coefficient 1. We can therefore apply the Integer Roots Test. Any possible integer root of P must be a factor of the constant term 24. The possible integer roots are 1, 2, 3, etc. as well as their negatives. The complete list can be written as $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm 24$. This gives a total of 14 possible integer roots! At most three of these can be actual roots. It is also possible that none of them are actual roots. They must be tested one by one using synthetic division (trial-and-error).

As in Example 1, the quadratic factor $x^2 + x - 20$ can be factored by using the techniques of High School algebra or by continuing to test the possible integer roots. The factored form is

$$P(x) = (x + 1)^2(x - 3)(x + 5)(x - 4)$$

Example 3: Completely factor the polynomial $P(x) = 6x^4 - 5x^3 + 26x^2 - 25x - 20$.

Solution: Because the leading coefficient is not equal to 1, we apply the Rational Roots Test. Suppose that $\frac{n}{d}$ is a rational root of P

Then the possible values of the numerator are $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$.

The possible values of the denominator are $\pm 1, \pm 2, \pm 3, \pm 6$.

We find the possible combinations (n divided by d) of the above numbers. The possible rational roots are $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20, \pm \frac{1}{2}, \pm \frac{5}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{5}{3}, \pm \frac{10}{3}, \pm \frac{20}{3}, \pm \frac{1}{6}, \pm \frac{5}{6}$.

We find that $k = \frac{1}{2}$ is a root of P .

$$\begin{array}{r|rrrrr} \frac{1}{2} & 6 & -5 & 26 & -25 & 20 \\ & & 3 & -4 & 15 & -20 \\ \hline & 6 & -8 & 30 & -40 & 0 \end{array} \quad ! \quad P(x) = \left(x + \frac{1}{2}\right) (6x^3 - 8x^2 + 30x - 40)$$

Working with the second factor we find that $k = \frac{4}{3}$ is a root.

$$\begin{array}{r|rrrr} \frac{4}{3} & 6 & -8 & 30 & -40 \\ & & 8 & 0 & 40 \\ \hline & 6 & 0 & 30 & 0 \end{array} \quad ! \quad P(x) = \left(x + \frac{1}{2}\right) \left(x - \frac{4}{3}\right) (6x^2 + 30)$$

The quadratic factor $6x^2 + 30$ can be rewritten $6(x^2 + 5)$. This quadratic factor can not be factored further. The final (simplified) factored form is

$$P(x) = 6 \left(x + \frac{1}{2}\right) \left(x - \frac{4}{3}\right) (x^2 + 5) \quad \text{which can be rewritten as} \quad P(x) = (2x + 1)(3x - 4)(x^2 + 5).$$

Exercises: Completely factor each of the following polynomials.

We continue by testing for roots in the second factor.

$$3 \begin{array}{r|rrrr} 1 & 2 & 23 & 60 \\ & 3 & 3 & 60 \\ \hline 1 & 1 & 20 & 0 \end{array} \quad ! \quad k = 3 \text{ is a root} \quad ! \quad P(x) = (x - 2)(x + 3)(x^2 - x - 20)$$

The third factor may be factored by the techniques of High School algebra.

$$P(x) = (x - 2)(x + 3)(x + 4)(x - 5)$$

2. The possible integer roots are $\pm 1, \pm 2, \pm 5, \pm 10$ (Integer Roots Test). We test these possibilities to find the actual roots of the polynomial.

$$1 \begin{array}{r|rrrr} 1 & 1 & 15 & 23 & 10 \\ & 1 & 2 & 13 & 10 \\ \hline 1 & 2 & 13 & 10 & 0 \end{array} \quad ! \quad k = 1 \text{ is a root} \quad ! \quad P(x) = (x - 1)(x^3 + 2x^2 - 13x + 10)$$

We continue testing for roots in the second factor. **As mentioned earlier, it is important to test the same root a second time. Roots can, and sometimes do, repeat.**

$$1 \begin{array}{r|rrrr} 1 & 1 & 2 & 13 & 10 \\ & 1 & 3 & 10 & 10 \\ \hline 1 & 3 & 10 & 0 & 0 \end{array} \quad ! \quad k = 1 \text{ is a root (again)} \quad ! \quad P(x) = (x - 1)^2(x + 5)(x - 2)$$

The third factor may be factored by the techniques of High School algebra.

$$P(x) = (x - 1)^2(x + 5)(x - 2)$$

3. The Rational Roots Test allows us to find the possible rational roots:

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}, \pm \frac{9}{4}$$

We test these possibilities to find the actual roots of the polynomial.

$$2 \begin{array}{r|rrrr} 4 & 9 & 25 & 36 & 36 \\ & 8 & 2 & 54 & 36 \\ \hline 4 & 1 & 27 & 18 & 0 \end{array} \quad ! \quad k = 2 \text{ is a root} \quad ! \quad P(x) = (x - 2)(4x^3 - x^2 - 27x + 18)$$

We continue by testing for roots in the second factor.

$$2 \begin{array}{r|rrrr} 4 & 1 & 27 & 18 \\ & 8 & 18 & 18 \\ \hline 4 & 9 & 9 & 0 \end{array} \quad ! \quad k = -2 \text{ is a root} \quad ! \quad P(x) = (x - 2)(x + 2)(4x^2 - 9x + 9)$$

The third factor may be factored by the techniques of High School algebra.

$$P(x) = (x - 2)(x + 2)(x - 3)(4x + 3)$$

4. The possible rational roots are 1, 2, 3, 4, 6, 8, 12, 24, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{4}{3}$, $\frac{8}{3}$.

We test these possibilities to find the actual roots of the polynomial.

$$1 \left| \begin{array}{cccccc} 3 & 10 & 11 & 46 & 4 & 24 \\ & 3 & 7 & 18 & 28 & 24 \\ \hline & 3 & 7 & 18 & 28 & 24 & 0 \end{array} \right. \quad ! \quad k = -1 \text{ is a root} \quad ! \quad P(x) = (x+1)(3x^4+7x^3-18x^2-28x+24)$$

We continue by testing for roots in the second factor.

$$2 \left| \begin{array}{cccccc} 3 & 7 & 18 & 28 & 24 \\ & 6 & 26 & 16 & 24 \\ \hline & 3 & 13 & 8 & 12 & 0 \end{array} \right. \quad ! \quad k = 2 \text{ is a root} \quad ! \quad P(x) = (x+1)(x-2)(3x^3+13x^2+8x-12)$$

We continue by testing for roots in the third factor.

$$2 \left| \begin{array}{cccc} 3 & 13 & 8 & 12 \\ & 6 & 14 & 12 \\ \hline & 3 & 7 & 6 & 0 \end{array} \right. \quad ! \quad k = -2 \text{ is a root} \quad ! \quad P(x) = (x+1)(x-2)(x+2)(3x^2+7x-6)$$

The fourth factor may be factored by the techniques of High School algebra.

$$P(x) = (x+1)(x-2)(x+2)(3x-2)(x+3)$$

Drill Exercises: Completely factor each of the following polynomials.

1. $x^3 + 5x^2 - 12x - 36$
2. $x^3 - 6x^2 - 7x + 60$
3. $x^4 + 9x^3 - 6x^2 - 116x + 168$
4. $x^5 - 6x^4 - 15x^3 + 80x^2 + 84x - 144$
5. $x^4 + 8x^3 + 5x^2 - 74x - 120$
6. $x^5 + 6x^4 - 28x^3 - 110x^2 - 117x - 40$
7. $2x^3 + x^2 - 85x + 42$
8. $24x^3 - 62x^2 + 31x + 12$
9. $20x^4 + 12x^3 - 33x^2 - 5x + 6$
10. $9x^5 + 30x^4 - 92x^3 - 498x^2 - 685x - 300$
11. $8x^4 - 20x^3 - 18x^2 + 81x - 54$
12. $72x^5 + 90x^4 - 383x^3 - 287x^2 + 254x - 40$

Answers to Drill Exercises:

1. $(x+6)(x+2)(x-3)$
2. $(x+3)(x-4)(x-5)$
3. $(x+7)(x+6)(x-2)^2$
4. $(x+3)(x+2)(x-1)(x-4)(x-6)$
5. $(x+5)(x+4)(x+2)(x-3)$
6. $(x+8)(x-5)(x-x)(x+7)(3x-4)($