

ALGEBRA MODULES (Revised Aug. 2000)

ALGEBRA MODULE ONE

INTEGRATION AND FACTORIZATION

LONG DIVISION OF POLYNOMIALS

x - 2.

$$\begin{array}{r} 2x^2 + 2x + 7 \\ x - 2 \overline{) 2x^3 - 2x^2 + 3x - 5} \\ \underline{2x^3 - 4x^2} \\ 2x^2 + 3x \\ \underline{2x^2 - 4x} \\ 7x - 5 \\ \underline{7x - 14} \\ 9 \end{array}$$

Example 2: $p(x) = x^3 - 3x^2 - x + 3$.

The possible rational zeros are the factors of 3 since the leading coefficient is 1. Possible zeros = $\pm 1, \pm 3$. It is easy to verify that $p(1) = 0$, so $x - 1$ is a factor. Using long division:

$$\begin{array}{r} x^2 - 2x - 3 \\ x-1 \overline{) x^3 - 3x^2 - x + 3} \\ \underline{x^3 - x^2} \\ -2x^2 - x \\ \underline{-2x^2 + 2x} \\ -3x + 3 \\ \underline{-3x + 3} \\ 0 \end{array}$$

$$p(x) = (x - 1)(x - 3)(x + 1)$$

SPECIAL FACTORIZATIONS

PERFECT SQUARES: $(a - b)^2 = a^2 - 2ab + b^2$ $(a + b)^2 = a^2 + 2ab + b^2$

Example 2B: (GROUPING and FACTORING)

$$3x^2 - 5x - 2$$

$$= 3x^2 - 6x + x - 2 \quad (\text{SPLIT the middle term.})$$

$$= 3x(x - 2) + 1(x - 2) \quad (\text{Grouping and Factoring})$$

$$= (3x + 1)(x - 2)$$

FOURTH DEGREE POLYNOMIALS: Fourth degree polynomials in the form $ax^4 + bx^2 + c$ may be factored as quadratics.

Example 3: $x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4) = (x - 1)(x + 1)(x - 2)(x + 2)$

Example 4: $x^4 + 6x^2 - 7 = (x^2 - 1)(x^2 + 7) = (x - 1)(x + 1)(x^2 + 7)$

PROBLEMS

A. In the following problems, one of the zeros of the polynomial is given. Use the Factor

and division to factor the polynomial completely

ANSWERS

- A. 1) $(x-1)(x+1)(x-4)$ 4) $x(x+1)(x-2)(x-8)$
2) $(x+3)^2(x-1)$ 5) $(x-2)(x^2-x+2)$
3) $(x-2)^2(x+1)$ 6) $(x+1)(2x-5)(x+2)$
- B. 1) $1 + \frac{4x+1}{x^2-1}$ 4) $x^2 - 2x + 7 - \frac{9}{x+2}$
2) $x - 2 + \frac{-x+3}{x^2+1}$ 5) $x - 4 + \frac{8x+7}{x^3+2x^2+x}$
3) $x^2 - 2x + 4 - \frac{6}{x+2}$ 6) $x + \frac{-7x^2+2}{x^3+4x}$
- C. 1) $(x-2)(x+2)(x^2+4)$ 9) $(x-1)(x+1)(x^2+2)$
2) $(x-2)(x^2+2x+4)$ 10) $(x-2)(x^2+4)$
3) $(x-5)(x+3)$ 11) $(x+1)^2(x-2)$
4) $x(x-6)(x+4)$ 12) $(x+1)(x-2)(x-3)$
5) $(3x+2)(2x-1)$ 13) $(x-3)(x-4)(x+2)$
6) $(4x+1)(2x-3)$ 14) $(x-1)(x-2)(x-6)$
7) $(x^2-1)(x^2-9) = (x-1)(x+1)(x-3)(x+3)$
8) $(2x+3)(4x^2-6x+9)$ 15) $(x+2)^3(x-3)$

ALGEBRA MODULE TWO

EXPONENTS AND RADICALS

REVIEW OF PROPERTIES OF EXPONENTS

Let x and v be real numbers, and let m and n be integers.

$$1) \quad x^m x^n = x^{m+n}$$

$$2^3 2^2 = 2^{3+2} = 2^5 = 32$$

$$2) \quad \frac{x^m}{x^n} = x^{m-n}$$

$$\frac{x^5}{x^2} = x^{5-2} = x^3$$

$$3) \quad x^0 = 1$$

$$3^0 = 1$$

$$4) \quad \frac{1}{y^n} = y^{-n}$$

$$\frac{1}{y^3} = y^{-3}$$

$$6) \quad (x^m)^n = x^{mn}$$

$$(x^2)^3 = x^6$$

$$7) \quad \left(\frac{x}{v}\right)^m = \frac{x^m}{v^m}$$

$$\left(\frac{5}{y}\right)^2 = \frac{25}{y^2}$$

FRACTIONAL EXPONENTS

When we write a fractional exponent such as $x^{m/n}$, we can use property 6) above to evaluate the term.

$$1) \quad 8^{2/3} = \left(8^{1/3}\right)^2 = 2^2 = 4$$

$$2) \quad 8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{4}$$

$$3) \quad (-8)^{2/3} = \left(-8^{1/3}\right)^2 = (-2)^2 = 4$$

$$4) \quad (-8)^{-2/3} = \frac{1}{(-8)^{2/3}} = \frac{1}{4}$$

RADICALS

A. Rewrite the expression to eliminate the radicals in the denominator.

$$1) \quad \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \quad (\text{this process is called rationalizing the denominator.})$$

For an expression with a sum or difference involving a radical, such as $x + b\sqrt{y}$ we can eliminate the radical in the denominator by multiplying the numerator and the denominator by the conjugate $x - b\sqrt{y}$. This is also called rationalizing. This process makes use of the difference of squares because

$$(x + b\sqrt{y})(x - b\sqrt{y}) = x^2 - b^2y$$

$$2) \quad \frac{4}{3 - \sqrt{2}} = \frac{4}{3 - \sqrt{2}} \cdot \frac{3 + \sqrt{2}}{3 + \sqrt{2}} = \frac{4(3 + \sqrt{2})}{9 - 2} = \frac{4(3 + \sqrt{2})}{7}$$

SOLVED PROBLEMS INVOLVING EXPONENTS

Simplify the following expressions

1) $(7^2 a^2 b^3)(7^4 a^9 b^2)$

Solution: Regrouping and combining like terms.

$$(7^2 a^2 b^3)(7^4 a^9 b^2) = (7^2 7^4)(a^2 a^9)(b^3 b^2) = 7^6 a^{11} b^5 \quad (\text{adding exponents})$$

2) $\frac{128x^3}{32x^5}$

Solution: We express 128 and 32 as powers of the common base 2.

$$\frac{128x^3}{32x^5} = \frac{2^7 x^3}{2^5 x^5} = 2^2 x^{-2} \quad (\text{subtracting exponents})$$

$$= \frac{4}{x^2} \quad \left(x^{-2} = \frac{1}{x^2} \right)$$

$(2x)^{-3}$

$$\text{So } \frac{(2x)^{-3}}{(3y)^{-2}} = \frac{(3y)^2}{(2x)^3} = \frac{3^2 y^2}{2^3 x^3} = \frac{9y^2}{8x^3}$$

4) $(3x^{-2}y^3)^4$

$$\text{Solution: } (3x^{-2}y^3)^4 = \left(\frac{3y^3}{x^2} \right)^4 = \frac{3^4 y^{12}}{x^8} = \frac{81y^{12}}{x^8}$$

6) $3^{2n+1} \cdot 3^{n-2}$

Solution: We add the exponents and combine like terms.

$$3^{2n+1} \cdot 3^{n-2} = 3^{(2n+1)+(n-2)} = 3^{3n-1}$$

7) $\frac{2^{4n+1}}{2^{5n-1}}$

Solution: We subtract the exponents and combine like terms.

$$\frac{2^{4n+1}}{2^{5n-1}} = 2^{(4n+1)-(5n-1)} = 2^{(4n-5n)+(1-(-1))}$$

$$= 2^{-n+2}$$

$$= 2^{2-n} \text{ or } \frac{1}{2^{n-2}}$$

8) $\frac{4^{n+3}}{2^{n-1}}$

Solution: Change 4 to 2^2 .

Thus

$$\begin{aligned} \frac{4^{n+3}}{2^{n-1}} &= \frac{(2^2)^{n+3}}{2^{n-1}} = \frac{2^{2n+6}}{2^{n-1}} = 2^{(2n+6)-(n-1)} \\ &= 2^{(2n-n)+(6-(-1))} \\ &= 2^{n+7} \end{aligned}$$

9) $\sqrt{50x^4y^8}$

Solution: Use the fact that $\sqrt{abc} = \sqrt{a} \cdot \sqrt{b} \cdot \sqrt{c}$

$$\begin{aligned} \sqrt{50x^4y^8} &= \sqrt{50} \sqrt{x^4} \sqrt{y^8} \\ &= \sqrt{25} \sqrt{2} \sqrt{x^4} \sqrt{y^8} \quad (\text{NOTE: } \sqrt{x^4} = \sqrt{(x^2)^2} = x^2 \\ &\qquad \qquad \qquad \sqrt{y^8} = \sqrt{(y^4)^2} = y^4) \\ &= 5\sqrt{2} x^2 y^4 \\ &= \sqrt{2}(5x^2y^4) \end{aligned}$$

10) $\sqrt[3]{128x^4y^7}$

Solution: Use the fact that $\sqrt[3]{abc} = \sqrt[3]{a} \sqrt[3]{b} \sqrt[3]{c}$.

Also $\sqrt[3]{x^3} = x$ $\sqrt[3]{y^6} = \sqrt[3]{(y^2)^3} = y^2$ and $128 = 2^7 = 2^6 \cdot 2^1$

So

$$\begin{aligned}\sqrt[3]{128x^4y^7} &= \sqrt[3]{128} \sqrt[3]{x^4} \sqrt[3]{y^7} \\ &= (\sqrt[3]{2^6} \sqrt[3]{2})(\sqrt[3]{x^3} \sqrt[3]{x})(\sqrt[3]{y^6} \sqrt[3]{y}) \\ &= (2 \sqrt[3]{2})(\sqrt[3]{x} \sqrt[3]{x})(\sqrt[3]{y^2} \sqrt[3]{y})\end{aligned}$$

$$= (2^2 xy^2) \sqrt[3]{x} \sqrt[3]{2} \sqrt[3]{y} = (4xy^2) \sqrt[3]{2xy}$$

Solve for k.

1) $3^{n+1} \cdot 3^k = 3^{n-2}$

Solution: $3^k = \frac{3^{n-2}}{3^{n+1}}$ (Divide by 3^{n+1}).

$3^k = 3^{-2-1} = 3^{-3}$. (The n's cancel out.)

Comparing exponents we see $k = -3$.

2) $\frac{2^{n+3}}{2^{k+1}} = 2^{2n-1}$

Solution: Subtracting exponents on the left side gives

Setting exponents equal and solving for k.

$$n - k + 2 = 2n - 1$$

$$(n + 2) - (2n - 1) = k$$

$$n + 3 - k$$

D. Simplify:

1) $(9x^5)(27x^2)$

2) $\frac{1024x^{-4}y^{-2}}{64x^{-6}y^3}$

3) $(3x^2y^{-3})^{-2}$

4) $\left(\frac{25x^{-3}}{625y^{-2}}\right)^{-2}$

5) $\frac{(2x^{-1}y^3)^3(4x^2y^{-3})^2}{(32x^{-5}y^4)^3}$

6) $\frac{(4x^{-3})^{-2}}{(5y^{-2})^{-4}}$

7) $\frac{5^{2n} \cdot 5^{n+1}}{5^{4n-2}}$

8) $\frac{3^{n+1} 9^{n-1}}{(27)^{n+2}}$

9) $\frac{(32)^{n+1}}{(64)^{n-1}}$

10) $\sqrt[3]{54x^5y^{13}}$

E. Solve for k.

1) $3^{n+2} 3^{k-1} = 3^{n-3}$

2) $\frac{2^{n+1}}{2^{k-1}} = 2^{3n}$

A. 1) 27

2) 9

3) $\frac{1}{16}$

4) 8

5) $\frac{1}{4}$

3/

2/

4/

5/

ALGEBRA MODULE THREE

RATIONAL FUNCTIONS

The properties for manipulations of fractions apply also to fractions of polynomials which are known as rational functions.

1) ADDITION: $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

2) MULTIPLICATION: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

3) DIVISION: $\frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ and also $\frac{a/b}{c} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$

1. Simplify the result as much as possible.

$$1 \quad 1 \quad \frac{x - (x+h)}{\dots}$$

PROBLEMS

Perform the indicated operation and write the answer in factored form where possible:

$$x \quad x-2$$

$$2 \quad 1$$

$$1 \quad 4$$

