DAWSON COLLEGE
Mathematics Department
Final Examination
Linear Algebra
201-NYC-05 Sections 01, 02, 03, 04, 05, 06
December 19th, 2017
9:30-12:30

Student Name Solutions

Student I.D. # ____________

Teacher ____________

Instructors:
B. Szczepara, T. Kengatharam, V. Ohanyan, Y. Lamontagne, S. Muise, A. Juhasz,

Question #	Mark
1	/5
2	/5
3	/5
4	/4
5	/6
6	/4
7	/8
8	/8
9	/4
10	/8
11	/4
12	/5
13	/12
14	/8
15	/4
16	/6
17	/4
Total	/100

Instructions:

- **Print your name and student ID number in the space provided above.**
- **All questions are to be answered directly on the examination paper in the provided space.**
- **Translation and regular dictionaries are permitted.**
- **The Sharp EL-531XG or EL-531X calculators are permitted.**
- **This examination consists of 17 questions.**
- **This exam booklet must be returned intact.**

1. *[4+1 marks]*

a) Determine the general solution of the following system using Gauss-Jordan Elimination:

$$\begin{cases} x_1 \qquad + 2x_3 + x_4 = -1 \\ 2x_1 + x_2 + 3x_3 + 3x_4 = 2 \\ -x_1 + x_2 - 3x_3 \qquad = 5 \\ 3x_1 + 2x_2 + 7x_3 + 2x_4 = -4 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 2 & 1 & 3 & 3 & 2 \\ -1 & 1 & -3 & 0 & 5 \\ 3 & 2 & 7 & 2 & -4 \end{array}\right] \xrightarrow[\substack{r_3 : r_3 + r_1 \\ r_4 : r_4 - 3r_1}]{r_2 : r_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 2 & 1 & -1 & -1 \end{array}\right] \xrightarrow[\substack{r_4 : r_4 - 2r_2 \\ (r_3 \leftrightarrow r_4)}]{r_3 : r_3 - r_2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 0 & 3 & -3 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_3 : \frac{1}{3} r_3} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[r_2 : r_2 + r_3]{r_1 : r_1 - 2r_3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 5 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\begin{aligned} X_1 &= 5 - 3t \\ X_2 &= 1 \qquad\qquad t \in \mathbb{R} \\ X_3 &= -3 + t \\ X_4 &= t \end{aligned}}$$

b) Find the particular solution when $x_1 = 2$

$$2 = 5 - 3t$$
$$-3 = -3t$$
$$\Rightarrow \boxed{t = 1}$$

$$\boxed{\begin{aligned} X_1 &= 2 \\ X_2 &= 1 \\ X_3 &= -2 \\ X_4 &= 1 \end{aligned}}$$

2. *[5 marks]* Determine the conditions on a such that the system has
a) no solution b) one solution c) infinitely many solutions:

$$x + 4y - 10z = 1$$
$$-3x + 2y + 2z = 2$$
$$4x + 2y + (a^2 - 13)z = a$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -10 & 1 \\ -3 & 2 & 2 & 2 \\ 4 & 2 & a^2-13 & a \end{array}\right] \xrightarrow[r_3 : r_3 - 4r_1]{r_2 : r_2 + 3r_1} \left[\begin{array}{ccc|c} 1 & 4 & -10 & 1 \\ 0 & 14 & -28 & 5 \\ 0 & -14 & a^2+27 & a-4 \end{array}\right] \xrightarrow{r_3 : r_3 + r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -10 & 1 \\ 0 & 14 & -28 & 5 \\ 0 & 0 & a^2-1 & a+1 \end{array}\right]$$

a) no sol'n : $a = 1$
b) one sol'n : $a \neq \pm 1$, $a \in \mathbb{R}$
c) infinitely many sol'ns : $a = -1$

3. *[3+2 marks]* Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 2 & 4 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

~~Evaluate (wherever possible):~~

a) $A^{-1}BC$

$$= -1 \underbrace{\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 2 & 4 & -1 \end{bmatrix}}_{2\times 4} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}_{4\times 4}$$

$$= -1 \begin{bmatrix} 5 & -14 & -3 & 17 \\ -3 & 8 & 1 & -10 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$= -1 \begin{bmatrix} 5 & -28 & -6 & 51 \\ -3 & 16 & 2 & -30 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} -5 & 28 & 6 & -51 \\ 3 & -16 & -2 & 30 \end{bmatrix}}$$

b) C^{-4}

$$= \begin{bmatrix} 1^4 & 0 & 0 & 0 \\ 0 & (1/2)^4 & 0 & 0 \\ 0 & 0 & (1/2)^4 & 0 \\ 0 & 0 & 0 & (1/3)^4 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/16 & 0 & 0 \\ 0 & 0 & 1/16 & 0 \\ 0 & 0 & 0 & 1/81 \end{bmatrix}}$$

4. *[4 marks]* Solve for X when:

$$\begin{bmatrix} -13 & 1 \\ 27 & 2 \end{bmatrix} X = \begin{bmatrix} -11 & \frac{1}{2} \\ 29 & 1 \end{bmatrix} X - \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\left(\begin{bmatrix} -13 & 1 \\ 27 & 2 \end{bmatrix} - \begin{bmatrix} -11 & 1/2 \\ 29 & 1 \end{bmatrix} \right) X = -\begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1/2 \\ -2 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} -2 & 1/2 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & -1/2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} = -\begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix}}$$

5. *[3+3 marks]*

a) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & -5 & 0 \\ 2 & 4 & 1 \end{bmatrix}$; express A^{-1} as a product of elementary matrices

$$\begin{bmatrix} 1 & 2 & 0 \\ -3 & -5 & 0 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{r_2 : r_2 + 3r_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{r_3 : r_3 - 2r_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 : r_1 - 2r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = I \therefore A^{-1} = E_3 E_2 E_1 = \boxed{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

b) Prove that if A is an invertible matrix and B has the same reduced row echelon form as A, then B is also invertible

A is invertible $\Rightarrow$ I is the reduced row echelon form of A

if B has the same reduced echelon form as A

$\Rightarrow$ I is the reduced row echelon form of B

$\Rightarrow$ B is invertible

6. *[4 marks]* Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & k & 1 \\ 2 & 4 & k \end{bmatrix}$.

Determine the values of k for which the homogeneous system $Ax = 0$ has only the trivial solution

trivial sol'n whenever $\begin{vmatrix} 1 & 2 & 0 \\ 1 & K & 1 \\ 2 & 4 & K \end{vmatrix} = 1(K^2-4) - 2(K-2) \neq 0$

$K^2 - 4 - 2K + 4 \neq 0$

$K(K-2) \neq 0$

trivial sol'n whenever: $\boxed{K \neq 0,\ K \neq 2,\ K \in \mathbb{R}}$

7. *[4+4 marks]* Let A, B, C be 4x4 matrices such that $det(A) = 2$; $det(B) = 3$; $det(C) = -1$

Evaluate the following:

a) $\det(2A^T(3B)^{-1}C^{22}) = \det(2A^T)\det[(3B)^{-1}]\det(C^{22})$

$= 2^4 \det(A) \cdot \frac{1}{\det(3B)} \cdot [\det(C)]^{22}$

$= 2^4 \cdot 2 \cdot \frac{1}{3^4 \det(B)} \cdot (-1)^{22}$

$= \frac{2^5}{3^4 \cdot 3} = \frac{2^5}{3^5} = \boxed{\frac{32}{243}}$

b) $\det(AB^{-1} - A\,adj(B))$

$= \det[A(B^{-1} - adj(B))] = \det(A)\det(B^{-1} - adj(B))$

$= 2\det(B^{-1} - \det(B) \cdot B^{-1})$

$= 2\det(B^{-1} - 3B^{-1})$

$= 2\det(-2B^{-1})$

$= 2(-2)^4 \det(B^{-1})$

$= \frac{2^5}{\det(B)} = \boxed{\frac{32}{3}}$

8. [5+3 marks] For the following system: $\begin{cases} x + 2y = 1 \\ x + 3y + z = -2 \\ 2x + 4y + 3z = 0 \end{cases}$

a) Find A^{-1} using $adj(A)$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix} \qquad adj(A) = \begin{bmatrix} 5 & -1 & -2 \\ -6 & 3 & 0 \\ 2 & -1 & 1 \end{bmatrix}^T ; \qquad det(A) = 3$$

$$\boxed{A^{-1} = \frac{1}{3} \begin{bmatrix} 5 & -6 & 2 \\ -1 & 3 & -1 \\ -2 & 0 & 1 \end{bmatrix}}$$

b) Use your answer from part a) to determine the solution to the system

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 & -6 & 2 \\ -1 & 3 & -1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 7 \\ -5 \\ 2 \end{bmatrix} \boxed{= \begin{bmatrix} 7/3 \\ -5/3 \\ 2/3 \end{bmatrix}}$$

9. *[4 marks]* For the following system, use Cramer's rule to solve for z only (no marks will be given if Cramer's rule is not used):

$$x - y + 4z = 10$$
$$-2x + y + z = 0$$
$$4x - y + 2z = 6$$

$$\det(A) = \begin{vmatrix} 1 & -1 & 4 \\ -2 & 1 & 1 \\ 4 & -1 & 2 \end{vmatrix} = 4(-2) - 1(3) + 2(-1) = -13$$

$$\det(A_z) = \begin{vmatrix} 1 & -1 & 10 \\ -2 & 1 & 0 \\ 4 & -1 & 6 \end{vmatrix} = 10(-2) - 0(3) + 6(-1) = -26$$

$$z = \frac{\det(A_z)}{\det(A)} = \frac{-26}{-13} = \boxed{2}$$

10. *[4+4 marks]* Let $A(2,0,1)$, $B(4,-1,2)$, $C(1,2,1)$ be three points in $\mathbb{R}^3$, determine the following:

a) the area of the triangle determined by the two vectors $(\overrightarrow{AB} \times \overrightarrow{AC})$ and $\overrightarrow{BC}$

$\vec{AB} = (2,-1,1)$, $\vec{AC} = (-1,2,0)$, $\vec{BC} = (-3,3,-1)$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ -1 & 2 & 0 \end{vmatrix} = (-2,-1,3)$$

$$(\vec{AB} \times \vec{AC}) \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & 3 \\ -3 & 3 & -1 \end{vmatrix} = (-8,-11,-9)$$

$$\|(\vec{AB} \times \vec{AC}) \times \vec{BC}\| = \sqrt{64+121+81} = \sqrt{266}$$

$$\text{Area}_\Delta = \boxed{\frac{\sqrt{266}}{2} \text{ units}}$$

a) the angle between $(\overrightarrow{AB} \times \overrightarrow{AC})$ and $\overrightarrow{BC}$

$$\text{angle} = \cos^{-1}\left(\frac{(\vec{AB} \times \vec{AC}) \cdot (\vec{BC})}{\|(\vec{AB} \times \vec{AC})\| \; \|\vec{BC}\|}\right) = \cos^{-1}\left(\frac{(-2,-1,3)\cdot(-3,3,-1)}{\sqrt{14} \cdot \sqrt{19}}\right)$$

$$= \cos^{-1}\left(\frac{0}{\sqrt{14}\sqrt{19}}\right) = \cos^{-1}(0) = \boxed{\pi \text{ radians or } 90°}$$

or
$\vec{AB} \times \vec{AC}$ is a normal to the plane containing points A, B & C ∴ orthogonal to $\vec{BC}$

~~11. [4 marks]~~

Let $\vec{s} = (\vec{v} - \vec{w}\,)$ and $\vec{t} = (\vec{u} \times \vec{v}) + (\vec{v} \times \vec{w}) + (\vec{w} \times \vec{u})$. Show that $\vec{s}$ and $\vec{t}$ are orthogonal

$\vec{s} \circ \vec{t} =$

$$(\vec{v} - \vec{w}\,) \bullet \left[(\vec{u} \times \vec{v}) + (\vec{v} \times \vec{w}) + (\vec{w} \times \vec{u}\,) \right]$$

$$= \vec{v} \circ (\vec{u} \times \vec{v}) + \vec{v} \circ (\vec{v} \times \vec{w}\,) + \vec{v} \circ (\vec{w} \times \vec{u})$$

$$- \left[\vec{w} \circ (\vec{u} \times \vec{v}) + \vec{w} \circ (\vec{v} \times \vec{w}\,) + w \circ (\vec{w} \times \vec{u}) \right]$$

$$= 0 + 0 + \vec{v} \circ (\vec{w} \times \vec{u}\,) - \left[w \circ (\vec{u} \times \vec{v}\,) + 0 + 0 \right]$$

$$= \vec{v} \circ (\dot{w} \times \vec{u}\,) - \dot{w} \circ (\vec{u} \times \vec{v}\,)$$

$$= \vec{v} \cdot (\vec{w} \times \vec{u}) + \vec{w} \circ (\vec{v} \times \vec{u}\,)$$

$$= \vec{v} \circ (\vec{w} \times \vec{u}) - \vec{v} (\vec{w} \times \vec{u}\,)$$

$$= 0$$

$\therefore$ vectors are orthogonal

12. *[3+2 marks]* Given the plane $-2x + 3y = -z - 17$, the point $A(4, -2, 2)$, and the line

$$\begin{cases} x = -1 - 2t \\ y = 4 - t \qquad t \in \mathbb{R} \\ z = -5 + 6t \end{cases}$$

a) Find the point of intersection between the line and the plane

$$-2(-1-2t) + 3(4-t) + (-5+6t) = -17$$

$$2 + 4t + 12 - 3t - 5 + 6t = -17$$

$$7t = -26 \qquad t = -\frac{26}{7}$$

$$x = -1 + \frac{52}{7} = \frac{45}{7}$$

$$y = 4 + \frac{26}{7} = \frac{54}{7}$$

$$z = -5 - \frac{156}{7} = -\frac{191}{7}$$

$$\boxed{\left(\frac{45}{7}, \frac{54}{7}, -\frac{191}{7}\right)}$$

b) Find the general equation of the plane containing the point A and parallel the given plane

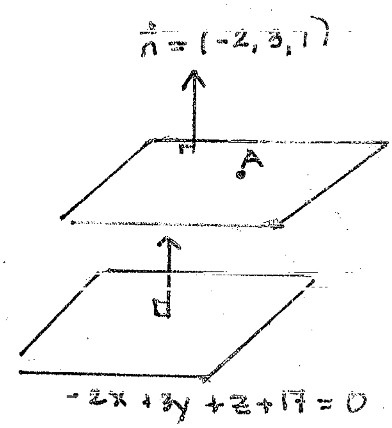
$$\vec{n} = (-2, 3, 1) \qquad A(4, -2, 2)$$

eq'n:

$$-2(x-4) + 3(y+2) + (z-2) = 0$$

$$-2x + 8 + 3y + 6 + z - 2 = 0$$

$$\boxed{-2x + 3y + z + 12 = 0}$$



13. *[4+4+4 marks]* Given the point $P(2,-6,8)$ and the lines:

$$l_1:\begin{cases} x=-2+3t \\ y=1 \\ z=3t \end{cases} \quad \text{and} \quad l_2:\begin{cases} x=s \\ y=-3s \\ z=3-2s \end{cases} \quad s,t\in\mathbb{R}$$

$\underline{\underline{\ell_1}}$: thru $R(-2,1,0)$ w/ directional vector $\vec{v}_1=(3,0,3)$

$\underline{\underline{\ell_2}}$: thru $Q(0,0,3)$ $\vec{v}_2=(1,-3,-2)$

a) compute the shortest distance between them

$$\begin{aligned} s-3t&=-2\\ -3s&=1\\ -2s-3t&=-3\end{aligned} \qquad \left[\begin{array}{cc|c} 1 & -3 & -2\\ -3 & 0 & 1\\ -2 & -3 & -3\end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -2\\ 0 & -9 & -5\\ 0 & -9 & -7\end{array}\right] \leftarrow \text{no sol'n}$$

$\therefore \ell_1$ & ℓ_2 do not intersect; (ii) $\vec{v}_1 \neq k\vec{v}_2$ $\therefore$ lines not parallel

$\Rightarrow \ell_1$ & ℓ_2 SKEW LINES

$$\vec{RQ}=(2,-1,3) \qquad \vec{v}_1\times\vec{v}_2=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k}\\ 3 & 0 & 3\\ 1 & -3 & -2\end{vmatrix}=(9,9,-9)=9\underbrace{(1,1,-1)}$$

let $\vec{n}=(1,1,-1)$

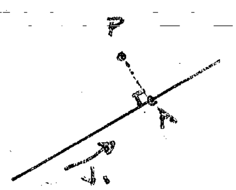
$$d=\|\text{proj}_{\vec{n}}\vec{RQ}\|=\frac{|\vec{n}\cdot\vec{RQ}|}{\|\vec{n}\|}=\frac{|(1,1,-1)\cdot(2,-1,3)|}{\sqrt{3}}=\frac{|-2|}{\sqrt{3}}=\boxed{\frac{2}{\sqrt{3}}\text{ units}}$$

~~b) find the point on l_1 that is closest to the point P~~

let A be the pt on ℓ_1 closest to P: $A(-2+3t,1,3t)$

$$\vec{PA}=(-2+3t-2,\ 1-(-6),\ 3t-8)=(-4+3t,\ 7,\ -8+3t)$$

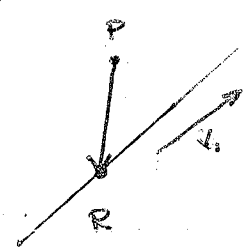
$$\vec{PA}\cdot\vec{v}_1=0:\ (-4+3t,\ 7,\ -8+3t)\cdot(3,0,3)=0$$
$$=-12+9t+0-24+9t=0$$
$$18t=36$$
$$t=2$$



$$A(-2+3(2),1,3(2))$$
$$\boxed{A(4,1,6)}$$

~~c) determine the general equation of the plane containing the line l_1 and the point P~~

$$\vec{v}_1=(3,0,3) \qquad \vec{PR}=(-2-2,\ 1-(-6),\ 0-8)=(-4,7,-8)$$



$$\vec{v}_1\times\vec{PR}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k}\\ 3 & 0 & 3\\ -4 & 7 & -8\end{vmatrix}=(-21,12,21)=3(-7,4,7)$$

let $\vec{n}=(-7,4,7)$

using P:

$$-7(x-2)+4(y+6)+7(z-8)=0$$
$$-7x+14+4y+24+7z-56=0$$
$$\boxed{-7x+4y+7z-18=0}$$

14. *[4 +4 marks]* Let V be the set of 2x2 skew symmetric matrices, i.e. $\mathcal{V} = \{A \mid A^T = -A\}$, with the usual operations of addition and scalar multiplication on matrices.

a) Show that $\mathcal{V}$ a subspace of $\boldsymbol{M}_{22}$ $\mathcal{V}$ not empty $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathcal{V}$

A1: let $B \in \mathcal{V}$, $(A+B)^T = A^T + B^T$

$= -A + (-B)$

$= -(A+B)$ $\therefore$ $\mathcal{V}$ closed under vector addition

S1: (or A6)

let $K \in \mathbb{R}$, $(KA)^T = KA^T = K(-A) = -K(A) = -(KA)$

$\therefore$ $\mathcal{V}$ closed under scalar multiplication

$\therefore$ $\mathcal{V}$ is a subspace of M_{22}

b) Determine a basis and calculate the dimension of the subspace $\mathcal{V}$

$$A \in \mathcal{V} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \Rightarrow \begin{matrix} a = d = 0 \\ b = -c \end{matrix}$$

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

$\boxed{\text{basis } \mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}}$ $\boxed{\dim(\mathcal{V}) = 1}$

15. *[4 marks]* Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ denote vectors in a vector space $\mathcal{V}$. Show that

$$span\{\vec{u},\vec{v},\vec{w}\} \subseteq span\{\vec{u}-\vec{v},\vec{u}+\vec{w},\vec{w}\}$$

need to show $\exists\ r, s, t \ni$

i) $\vec{u} = r(\vec{u}-\vec{v}) + s(\vec{u}+\vec{w}) + t(\vec{w}) \Rightarrow r=0,\ s=1,\ t=-1$

ii) $\vec{v} = r(\vec{u}-\vec{v}) + s(\vec{u}+\vec{w}) + t(\vec{w}) \Rightarrow r=-1,\ s=1,\ t=-1$

iii) $\vec{w} = r(\vec{u}-\vec{v}) + s(\vec{u}+\vec{w}) + t(\vec{w}) \Rightarrow r=0,\ s=0,\ t=1$

$\therefore$ span $\{\vec{u},\vec{v},\vec{w}\} \subseteq$ span $\{\vec{u}-\vec{v},\vec{u}+\vec{w},\vec{w}\}$

16. *[3 +3 marks]* Let A be the coefficient matrix of a **homogeneous** system.

$$A = \begin{bmatrix} 1 & 6 & 4 & 1 & 4 & 3 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix}$$

a) Find the solution to the system.

$$\xrightarrow{r_1: r_1 - 6r_2} \begin{bmatrix} 1 & 0 & -2 & -5 & 4 & -15 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1: r_1 - 4r_3} \begin{array}{c} \begin{array}{cccccc} & & r & s & & t \end{array} \\ \begin{bmatrix} 1 & 0 & -2 & -5 & 0 & -35 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$X_1 = 2r + 5s + 35t$

$X_2 = -1r - 1s - 3t$

$X_3 = r$

$X_4 = s$

$X_5 = -5t$

$X_6 = t$

Vector eq'n

$X = r(2,-1,1,0,0,0) + s(5,-1,0,1,0,0) + t(35,-3,0,0,$

b) Find a basis and the dimension for solution space.

$\boxed{B = \{(2,-1,1,0,0,0),\ (5,-1,0,1,0,0),\ (35,-3,0,0,-5,1)\}}$

$\boxed{\text{dimension} = 3}$

17. *[4marks]* For which values of $k \in \mathbb{R}$ is the following linearly independent in P_2

$$\{1+x, 3x+x^2, 2+x-kx^2\}$$

$P_2 : a_0 + a_1 x + a_2 x^2 \quad a_0, a_1, a_2 \in \mathbb{R}$

$r(1+x) + s(3x+x^2) + t(2+x-kx^2) = 0 + 0x + 0x^2$

$r + 0s + 2t = 0$ (coefficients of x^0)
$r + 3s + t = 0$ (coefficients of x^1)
$0r + s - k = 0$ (coefficients of x^2)

trivial sol'n (hence linearly independent in P_2)
as long as $\det(A) \neq 0$

where $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 1 \\ 0 & 1 & -k \end{bmatrix}$

$\det(A) = 1(-3k-1) + 2(1)$
$= -3k + 2$
$\neq 0$ if $\boxed{k \neq \frac{3}{2}, \; k \in \mathbb{R}}$