The Riemann Sum and the Definite Integral

We begin our introduction to the Riemann Sum by considering non-negative functions which are continuous over an interval $[a, b]$. To simplify the explanation and the calculations, the interval $[a, b]$ will be divided into subintervals of equal width, and the sample points will correspond to the right endpoints of the subintervals. A more general/rigorous treatment of the Riemann Sum may be found in the calculus textbook used by Pure and Applied Science students.

Let the non-negative function

Δ and where $x_n = b$. For each subinterval we construct a rectangle as shown in the diagram.

The base of each rectangle is Δx. The height of rectangle k (the rectangle on the subinterval with x_k as right endpoint) is $f(x_k)$. It follows that the area of rectangle k is $f(x_k)\Delta x$. The sum of the areas of all n rectangles is called the Riemann Sum. I.e. the Riemann Sum is equal to the expression $\sum_{k=1}^{n} f(x_k)\Delta x$. We see that the Riemann Sum is an approximation of the exact area under the graph of f from a to b. The larger the value of n the better the approximation. It can be proven that the limit at infinity of the Riemann Sum is the exact area under the graph of f from a to b. This limit has a special name and notation. It is called the definite integral.

Definition of Definite Integral If f is a continuous function defined on $[a, b]$, and if $[a, b]$ is divided into n equal subintervals of width $\Delta x = \frac{b-a}{n}$, and if $x_k = a + k\Delta x$ is the right endpoint of subinterval k, then the definite integral of f from a to b is the number

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

Note: In the following two examples we consider non-negative functions on the interval $[a,b]$. As explained last page, in such cases the definite integral from a to b is the area under the curve from a to b (i.e. the area between the curve and the x-axis). The summation formulas in the appendix will be needed in the solutions of these examples.

Example 1 Use the definition of definite integral to evaluate $\int_0^4 (2x^2 + 3)\,dx$.

We subdivide the interval $[0,4]$ into n equal subintervals of width $\Delta x = \dfrac{4-0}{n} = \dfrac{4}{n}$.

Then $x_k = 0 + k\left(\dfrac{4}{n}\right) = \dfrac{4k}{n}$ and $f(x_k) = 2\left(\dfrac{4k}{n}\right)^2 + 3 \rightarrow f(x_k) = \dfrac{32k^2}{n^2} + 3$.

$$f(x_k)\Delta x = \left(\frac{32k^2}{n^2} + 3\right)\left(\frac{4}{n}\right) = \frac{128k^2}{n^3} + \frac{12}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(\frac{128k^2}{n^3} + \frac{12}{n}\right) = \frac{128}{n^3}\sum_{k=1}^{n} k^2 + \frac{12}{n}\sum_{k=1}^{n} 1 = \frac{128}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{12}{n}\cdot n$$

$$= \frac{128}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) + 12 = \frac{64}{3}\left(1\right)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 12$$

$$\int_0^4 (2x^2+3)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\left[\frac{64}{3}\left(1\right)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 12\right] = \frac{64}{3}(1)(2) + 12 = \frac{164}{3}$$

Example 2 Use the definition of definite integral to evaluate $\int_2^5 (8x - x^2)\,dx$.

Until now we have only considered non-negative functions on the interval> @

2. $\frac{}{}$ $\frac{}{}$ $\frac{2}{}$ 10 $\frac{4}{}$ 10

6. $\Delta x = \frac{4}{n} \to x_k = 1 + \frac{4k}{n}$ and $f(x_k) = -\left(1 + \frac{4k}{n}\right) + 4\left(1 + \frac{4k}{n}\right)^2 \to f(x_k) = \frac{64k^2}{n^2} + \frac{28k}{n} + 3$

$f(x_k)\Delta x = \left(\frac{64k^2}{n^2} + \frac{28k}{n} + 3\right)\left(\frac{4}{n}\right) = \frac{256k^2}{n^3} + \frac{112k}{n^2} + \frac{12}{n}$

$\sum_{k=1}^{n} f(x_k)\Delta x = \frac{256}{n^3}\sum_{k=1}^{n} k^2 + \frac{112}{n^2}\sum_{k=1}^{n} k + \frac{12}{n}\sum_{k=1}^{n} 1 = \frac{256}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + \frac{112}{2}\left(1 + \frac{1}{n}\right) + 12$

$\frac{}{3}(1)(2) + 56(1) + 12 = \frac{460}{3}$

10. ' $\frac{4}{}$ o $\frac{4}{}$ 5 and $\frac{§4}{©}$ 5 $\cdot^{2}_{,1}$ 3 $\frac{§4}{©}$ 5 $\cdot_{,1}$ 5 o $\frac{16^{2}}{2}$ $\frac{28}{n}$ 15

APPENDIX

The following are useful formulas for working with summation notation.

1. $\sum_{k=1}^{n} c = nc$

2. $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

3. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

4. $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

5. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

6. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

7. $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$